Math 138 – Practice Final exam

Instructions:

- You have 3 hours to complete this exam.
- No external resources are allowed.
- Do not hesitate to ask for clarification on exam questions.

Question 1. (15 pts)

Please provide a **counterexample and/or disproof** to each of the following incorrect claims—be sure to justify why it is a counterexample/disproof.

1. **(5 pts)** If $(X, \leqslant)$ is a poset with $\#X \leqslant 2^n$, then $(X, \leqslant)$ admits an embedding into $(\mathcal{P}(\{1, \ldots, n\}), \subseteq)$.

2. **(5 pts)** For $a, b \in \mathbb{Z}$ and p a prime number, one has that $v_p(a+b) = \max(v_p(a), v_p(b))$.

3. **(5 pts)** Let $f\colon X \to Y$ be a function and consider $A \subseteq Y$. Then, we have the following equality of subsets of Y
$$A = f(f^{-1}(A)).$$

Question 2. (10 pts)

Prove by induction that 6 divides $n^3 - n$ for all $n \geqslant 1$.

Question 3. (15 pts)

Let n and m be coprime elements of $\mathbb{N}$ (i.e., they have no common prime divisors). Show that if $x \in \mathbb{N}$ and $\sqrt[m]{x^n}$ is rational, then $\sqrt[m]{x}$ is already rational.

Question 4. (15 pts)

Let $(X, \leqslant)$ be a poset. Define the *opposite poset* $(X, \leqslant^{\mathrm{op}})$ to have the same underlying set X, but with relation defined by

$$x \leqslant^{\mathrm{op}} y \Longleftrightarrow y \leqslant x.$$

1. **(7 pts)** Show that $(X, \leqslant^{\mathrm{op}})$ is a poset.
2. **(8 pts)** Show that if $(Y, \preceq)$ is another poset, then $(X, \leqslant) \simeq (Y, \preceq)$ if and only if $(X, \leqslant^{\mathrm{op}}) \simeq (Y, \preceq^{\mathrm{op}})$. (Recall: $\simeq$ means isomorphism)

Question 5. (20 pts)

Let $f\colon X \to Y$ be a function. Show that the following are equivalent:

1. f is bijective,
2. for all subsets $A \subseteq X$ the following equality of subsets of Y holds:

$$f(X - A) = Y - f(A).$$

Question 6. (25 pts)

Let $\mathbb{Q}$ be the set of rational numbers. Consider the set $\mathcal{I}$ of all closed intervals with rational endpoints:

$$\mathcal{I} = \{\, [a,b] : a, b \in \mathbb{Q},\ a \le b \,\}.$$

In the following, you are free to use any facts we proved in class (although state clearly those that you are using).

1. **(10 pts)** Prove that $\mathcal{I}$ is countable.

2. **(10 pts)** Let $\mathcal{U}$ be the set of all finite unions of such intervals, i.e.,

$$\mathcal{U} = \{\, [a_1, b_1] \cup \cdots \cup [a_n, b_n] \ :\ n \ge 1,\ [a_i, b_i] \in \mathcal{I} \text{ for each } i \,\}.$$

   Prove that $\mathcal{U}$ is countable.

3. **(5 pts)** Briefly explain why this does *not* contradict the fact that there are uncountably many subsets of $[0, 1]$.